San Diego Math Circle - Fermat
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Fuggedaboutit! An Introduction to Taxicab Geometry

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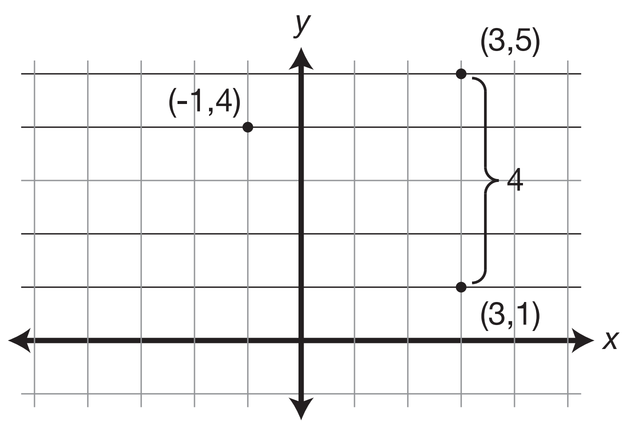
Introduction

Cartesian Coordinate System

In the Cartesian[1] coordinate system, points in a plane are uniquely specified by a pair of numbers, which specify the distances to two perpendicular axes, usually denoted x and y. The point (x, y) is x units away from the vertical y-axis, and y units away from the horizontal x-axis.

Distance

The distance between two coordinates is simple to calculate if they share the same x or y coordinate. For example, for the points $(3, 1)$ and $(3, 5)$, the distance between them is simply the displacement in the y direction, which is $|1 - 5| = |5 - 1| = 4$:



Pythagorean Theorem

In a right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs.

Triangle Conventions

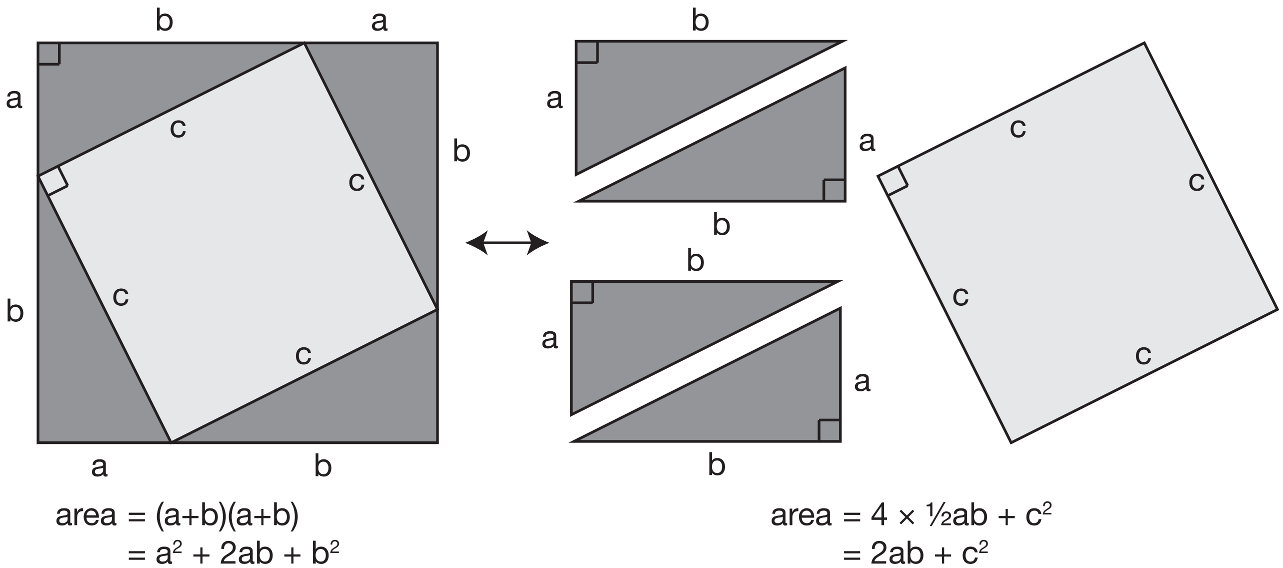
A (very) common convention is to name a triangle by its vertices. In triangle ABC, the vertices are A, B, and C, and the lengths of the sides are a, b, c, where a is the length of the side opposite vertex A, b is the length of the side opposite vertex B, and c is the length of the side opposite vertex C.

In a right triangle, it is typical for C to be the vertex with the right angle, and the **Pythagorean theorem** is then written as:

$$a^2 + b^2 = c^2.$$

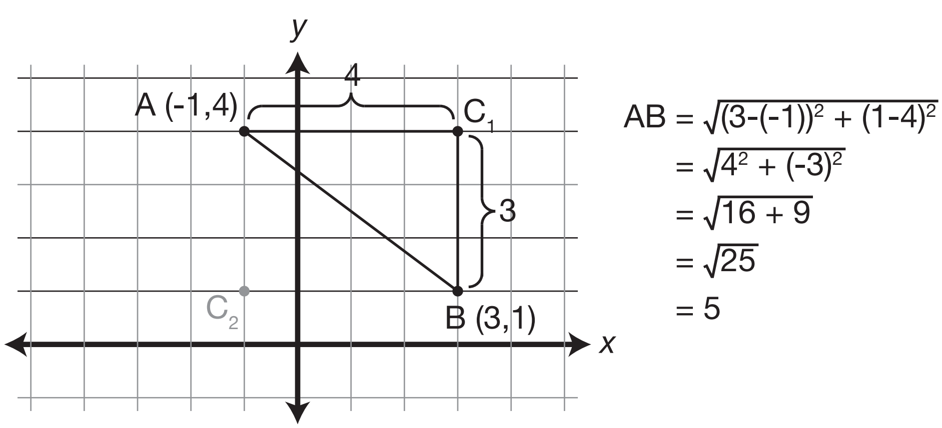
[1] named after 17th century French mathematician René Descartes

Proof



Euclidean Distance

The **Pythagorean theorem** now gives us a way to calculate the distance between two arbitrary points in the coordinate plane: we can create a right triangle with A at one of the points, and B at the other point, and then place C at either of two possible locations.



And then the distance is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. (We have dropped the absolute value signs because $|x_1 - x_2|^2 = (x_1 - x_2)^2 = (x_2 - x_1)^2$.)

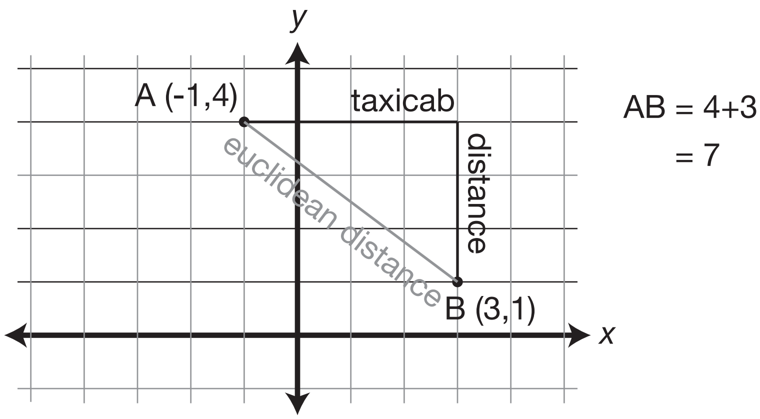
This standard definition of distance is called the Euclidean[2] norm, but it is not the only way to define distances...

Taxicab Geometry

The **taxicab distance** between two points in the plane, (x_1, y_1) and (x_2, y_2), is defined as $|x_1 - x_2| + |y_1 - y_2|$. In contrast to the **Euclidean distance** ("as the crow flies"), the **taxicab distance**

[2]named after 300 BC Greek mathematician Euclid

can be thought of as the distance that a taxicab would need to travel if driving on a Manhattan-like rectangular grid. (Manhattan distance is another common name.)

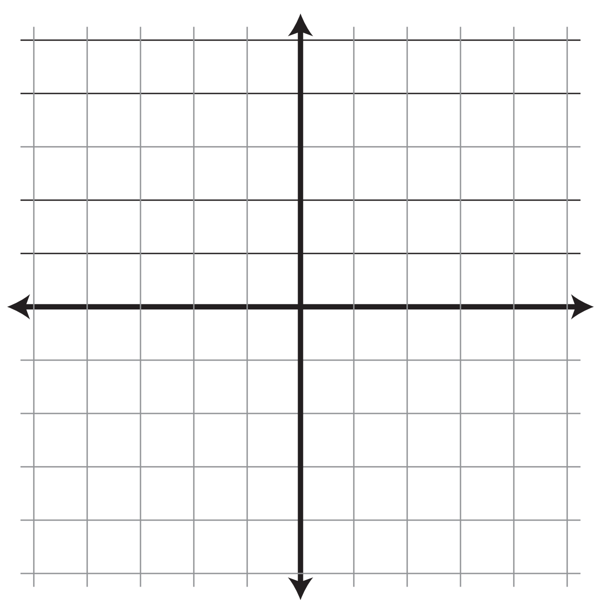


Circles

A **circle** is defined as the set of points that are the same distance from a fixed point, the center of the circle. This distance is the radius, r.

We can define a **taxicab circle** as the set of points that are the same **taxicab distance** from a fixed point, the center of the circle.

What does a **taxicab circle**, centered at the origin $(0, 0)$, with radius 2 look like?



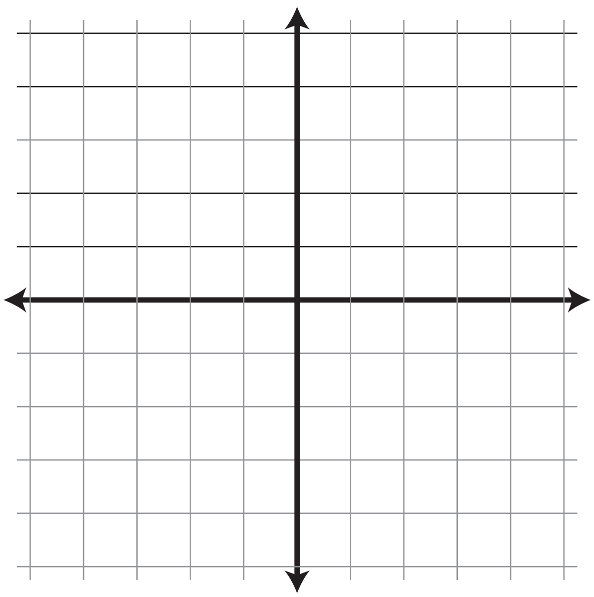
Triangles

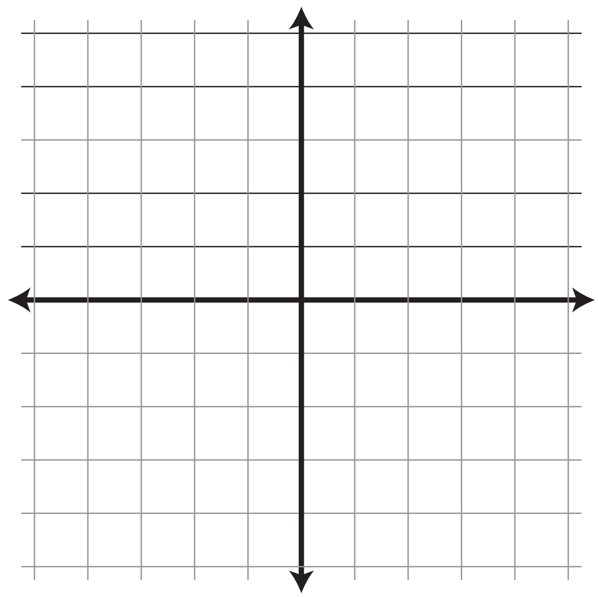
An **equilateral triangle** has three sides of the same length, and three equal angles, each $60°$.

If a **taxicab equilateral triangle** is a triangle with three sides with the same **taxicab length**, what might it look like? Can it also have three equal angles of $60°$ each?

Find the third point of the **taxicab equilateral triangle** whose other two points are $(0, 0)$ and $(2, 0)$.

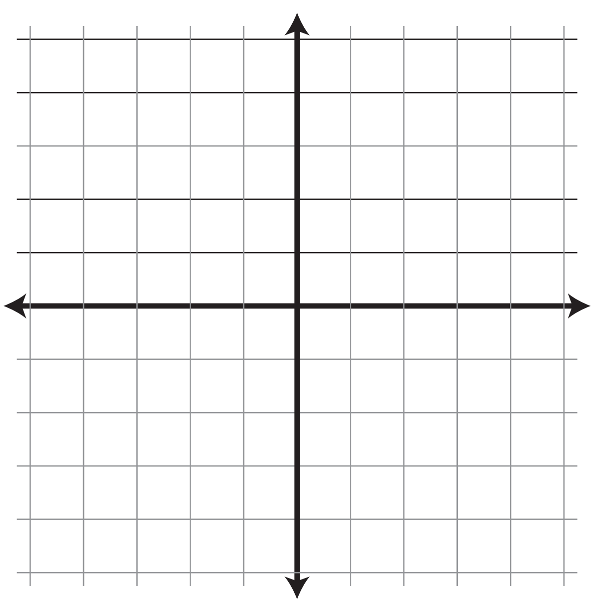
Find the third point of the **taxicab equilateral triangle** whose other two points are $(0, 0)$ and $(3, 1)$.

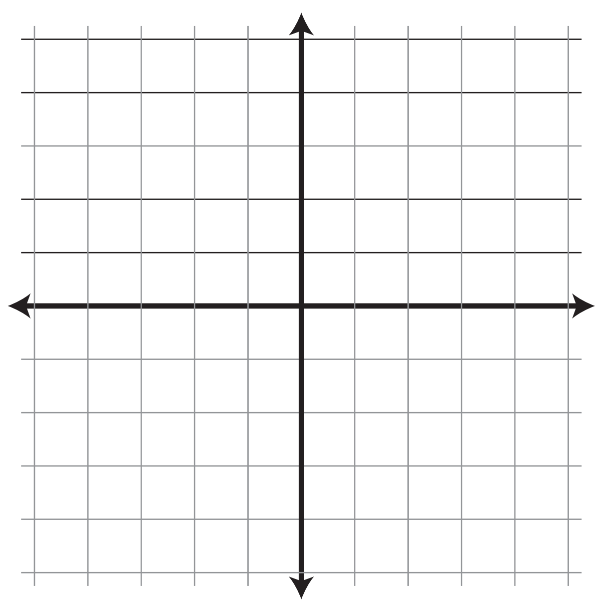




Triangle Inequality

The **triangle inequality** states that the sum of the lengths of any two sides of a triangle is greater than the length of the third side (for all triangles). Is this still true if **taxicab length** is used instead? Under what situations is it true or untrue?

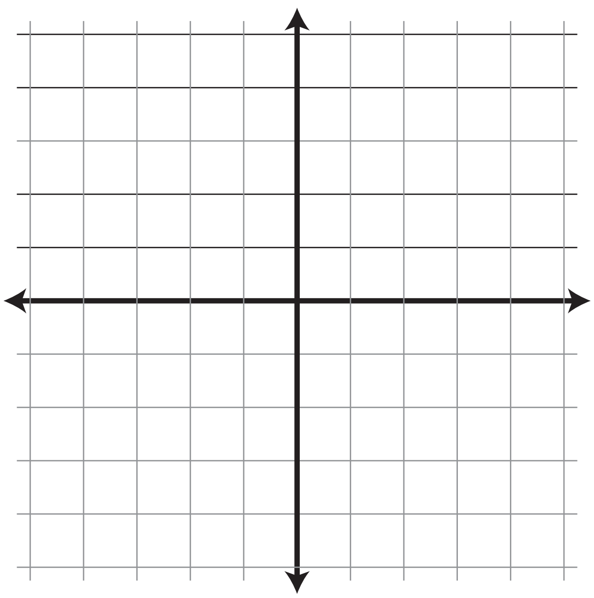




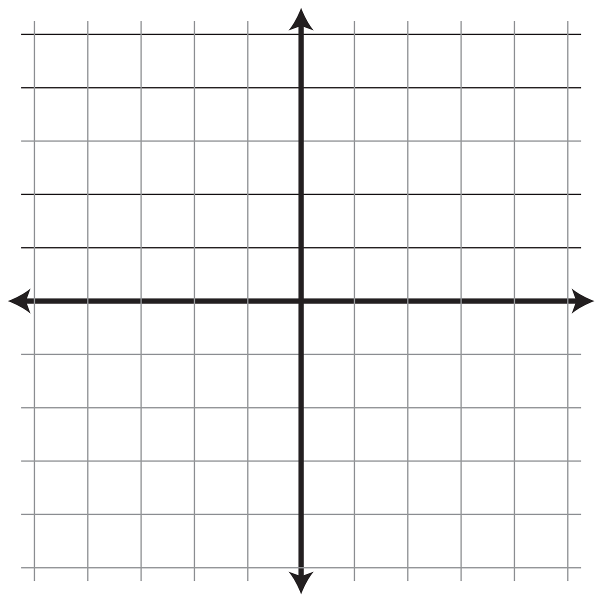
Equidistant Lines

Suppose we wanted to find the set of points that are always the same distance to two points A and B. For example, let $A = (0,0)$ and $B = (2,0)$. Then the set of points that are **equidistant** to both A and B is the line $x = 1$. It turns out that this line will always be the perpendicular bisector of the line segment $\overline{AB}$. What happens if we use **taxicab distance**?

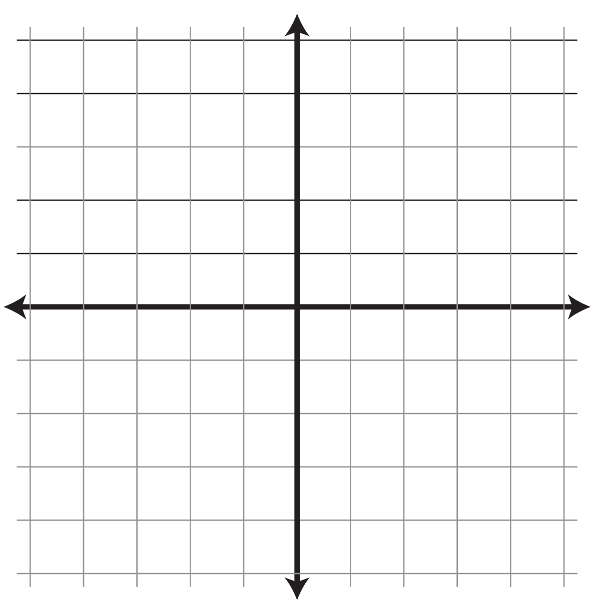
Find the set of points that are the same **taxicab distance** to $(0,0)$ and $(2,0)$.



Find the set of points that are the same **taxicab distance** to $(0,0)$ and $(3,1)$.



Find the set of points that are the same **taxicab distance** to $(0,0)$ and $(1,1)$.

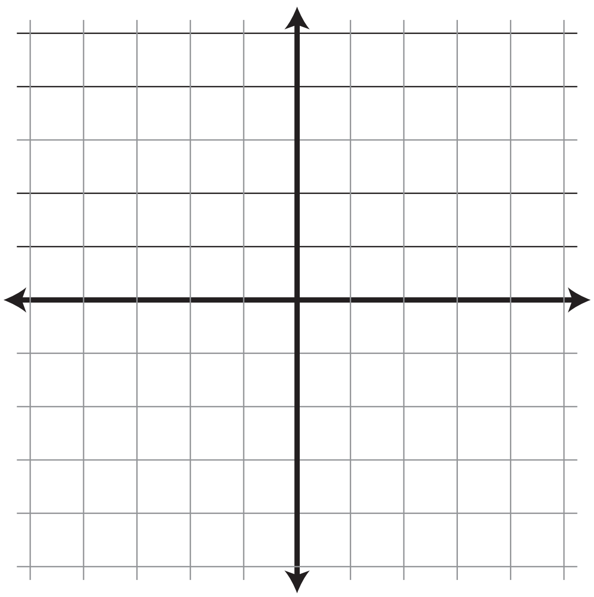


Ellipses

An ellipse is the set of points for which the sum of the distances to two points is the same. The two points are known as the **foci** (plural of focus). A circle is a special form of ellipse where the two foci are the same point, the center of the circle.

In a **taxicab ellipse**, the sum of the **taxicab distances** to the foci is the same.

Find the set of points for which the sum of the **taxicab distances** to $(0, 0)$ and $(2, 0)$ is equal to 4.



Find the set of points for which the sum of the **taxicab distances** to $(0, 0)$ and $(3, 1)$ is equal to 6.